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著者	八巻　俊輔
journal or publication title	IEEE Signal Processing Letters
volume	15
page range	465-468
year	2008
URL	http://hdl.handle.net/10097/47957

doi: 10.1109/LSP.2008.921479

Explicit Expressions of Balanced Realizations of Second-Order Digital Filters With Real Poles

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***Abstract*—This letter presents explicit expressions of the balanced realization of second-order digital filters with real poles. We consider two cases of second-order digital filters: that of real and distinct poles and that of real and multiple poles. Simple formulas are derived for the synthesis of the balanced realizations of these second-order digital filters.**

***Index Terms*—Balanced realizations, explicit expressions, state–space digital filters.**

I. Introduction

BALANCED realization is an important practical filter structure in the field of digital signal processing [1]–[4], which enables a balance between controllability and observability. This property gives digital filters many significant advantages. Balanced realizations have already been shown to have minimum statistical coefficient sensitivity and to be free of limit cycles [1], and it can be adopted as an initial realization to minimize the L_2-sensitivity of state–space digital filters [2]. Furthermore, model order reduction techniques based on balanced realizations have been investigated [3], [4]. These techniques can be applied not only to digital filters but also to analog filters. Therefore, balanced realizations have been widely used in many fields, such as digital signal processing, analog signal processing, and system control.

In general, balanced realizations are synthesized by using the algorithm in [3]. Although the method therein presented is a procedure for obtaining a balanced realization, it does not give the explicit expression of the filter coefficients. Such explicit expressions of the balanced realization would be very convenient since they would enable the synthesis of balanced realizations directly from the transfer function, thus greatly simplifying the synthesis procedure. The authors in [5] derived an explicit expression of the balanced realization of second-order digital filters with *complex conjugate poles*. However, the explicit expression given in [5] is not applicable to second-order digital filters with *real poles*. In other words, the conventional expression cannot cover all types of second-order digital filters. Actually, second-order digital filters with real poles cover a large region of the stability triangle, as shown in Fig. 1. Therefore, it is also necessary to derive the explicit expression of the balanced realization for second-order digital filters with *real poles*.

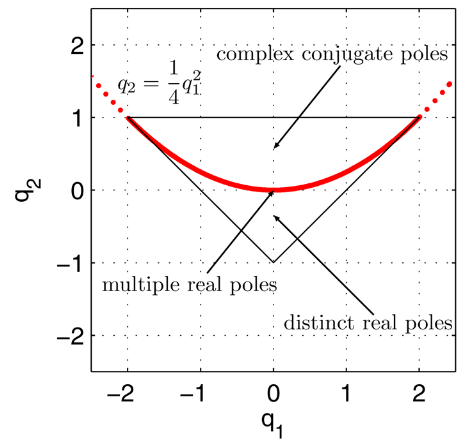

Fig. 1. Stability triangle of second-order digital filters.

To the best of our knowledge, there have been no attempts to give an explicit expression for the balanced realization of second-order digital filters with *real poles*. In this letter, we present explicit expressions of the balanced realizations of second-order digital filters with *real poles*.

II. Preliminaries

A. Second-Order Digital Filters

Consider a stable second-order IIR digital filter given by

$$H(z) = \frac{p_0 + p_1 z^{-1} + p_2 z^{-2}}{1 + q_1 z^{-1} + q_2 z^{-2}}. \qquad (1)$$

It is well known that the second-order digital filter $H(z)$ is stable, if and only if q_1 and q_2 remain within the stability triangle described by

$$|q_2| < 1, \quad |q_1| < 1 + q_2. \qquad (2)$$

Fig. 1 shows the stability triangle. For stable second-order digital filters given by (1), the locations of the poles depend on the filter coefficients q_1 and q_2 as follows:

$$\begin{cases} \text{Poles are complex conjugate if } q_1^2 - 4q_2 < 0. \\ \text{Poles are real and distinct if } q_1^2 - 4q_2 > 0. \\ \text{Poles are real and multiple if } q_1^2 - 4q_2 = 0. \end{cases} \qquad (3)$$

We synthesize the balanced realization by the state–space approach. The second-order digital filter (1) can be described by the following state–space representation:

$$\boldsymbol{x}(n+1) = \boldsymbol{A}\boldsymbol{x}(n) + \boldsymbol{b}u(n) \qquad (4)$$

$$y(n) = \boldsymbol{c}\boldsymbol{x}(n) + du(n) \qquad (5)$$

Manuscript received December 10, 2007; revised February 12, 2008. The associate editor coordinating the review of this manuscript and approving it for publication was Prof. Vesa Valimaki.

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Digital Object Identifier 10.1109/LSP.2008.921479

where $\boldsymbol{x}(n) = [x_1(n) \quad x_2(n)]^T$ is a state-vector, $u(n)$ is a scalar input, $y(n)$ is a scalar output, and $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, d)$ are coefficient matrices, described by

$$\left[\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{b} \\ \hline \boldsymbol{c} & d \end{array}\right] = \left[\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ \hline c_1 & c_2 & d \end{array}\right]. \tag{6}$$

The transfer function $H(z)$ is described in terms of the coefficient matrices $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, d)$ as $H(z) = \boldsymbol{c}(z\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b} + d$. The controllability Gramian $\boldsymbol{K}_0$ and the observability Gramian $\boldsymbol{W}_0$ are solutions to the following Lyapunov equations:

$$\boldsymbol{K}_0 = \boldsymbol{A}\boldsymbol{K}_0\boldsymbol{A}^T + \boldsymbol{b}\boldsymbol{b}^T \tag{7}$$

$$\boldsymbol{W}_0 = \boldsymbol{A}^T\boldsymbol{W}_0\boldsymbol{A} + \boldsymbol{c}^T\boldsymbol{c}. \tag{8}$$

The Gramians play important roles in the synthesis of high-accuracy state–space digital filters. When we synthesize digital filters which have low-sensitivity and low-roundoff noise, the Gramians are quite significant factors since both coefficient quantization error and roundoff error are formulated by using them. In the field of digital signal processing, the controllability and observability Gramians are also called covariance and noise matrices, respectively.

Let $\boldsymbol{T}$ be a nonsingular 2×2 real matrix. If a coordinate transformation defined by $\overline{\boldsymbol{x}}(n) = \boldsymbol{T}^{-1}\boldsymbol{x}(n)$ is applied to a filter realization $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, d)$, we obtain a new realization which has the following coefficient matrices

$$(\overline{\boldsymbol{A}}, \overline{\boldsymbol{b}}, \overline{\boldsymbol{c}}, \overline{d}) = (\boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T}, \boldsymbol{T}^{-1}\boldsymbol{b}, \boldsymbol{c}\boldsymbol{T}, d) \tag{9}$$

and the following Gramians:

$$(\overline{\boldsymbol{K}}_0, \overline{\boldsymbol{W}}_0) = \left(\boldsymbol{T}^{-1}\boldsymbol{K}_0\boldsymbol{T}^{-T}, \boldsymbol{T}^T\boldsymbol{W}_0\boldsymbol{T}\right) \tag{10}$$

respectively. It should be noted that the coordinate transformation does not affect the transfer function $H(z)$. In other words, the transfer function $H(z)$ is invariant under the coordinate transformations.

By the coordinate transformation, we can synthesize the balanced realization $(\boldsymbol{A}_\mathrm{b}, \boldsymbol{b}_\mathrm{b}, \boldsymbol{c}_\mathrm{b}, d_\mathrm{b})$, which is the filter structure whose controllability Gramian $\boldsymbol{K}_\mathrm{b}$ and observability Gramian $\boldsymbol{W}_\mathrm{b}$ are equal and diagonal as follows:

$$\boldsymbol{K}_\mathrm{b} = \boldsymbol{W}_\mathrm{b} = \mathrm{diag}(\theta_1, \theta_2) \tag{11}$$

where the parameters (θ_1, θ_2) are termed the second-order modes of the filter $H(z)$. The balanced realization, the Gramians of which are diagonalized, is used for the canonical form of the minimum statistical sensitivity realization [1] and the original form in the model order reduction techniques [3], [4].

III. Synthesis of the Balanced Realization of Second-Order Digital Filters With Real Poles

For second-order digital filters with *complex conjugate poles*, an explicit expression of the balanced realization is given in [5]. However, the explicit expression is not applicable to second-order digital filters with *real poles* since the authors in [5] define the transfer function of second-order digital filters as follows:

$$H(z) = \frac{\alpha}{z - \lambda} + \frac{\alpha^*}{z - \lambda^*} + d. \tag{12}$$

In this section, we propose an explicit expression of the balanced realization of second-order digital filters with *real poles*. The derivation method substantially differs from the method in [5] since $H(z)$ in (12) does not support the case of *real poles*.

A. Two Poles Are Real and Distinct $\left(q_1^2 - 4q_2 > 0\right)$

We consider second-order digital filters whose poles are real and distinct as follows:

$$H(z) = \frac{\alpha_1}{z - \lambda_1} + \frac{\alpha_2}{z - \lambda_2} + d \quad (\lambda_1 \neq \lambda_2) \tag{13}$$

where (λ_1, λ_2) are *real* poles, and (α_1, α_2) are *real* scalars. We define the scalar parameters P_1, P_2, and P_{12} as follows:

$$P_1 = \frac{|\alpha_1|}{1 - \lambda_1^2}, \quad P_2 = \frac{|\alpha_2|}{1 - \lambda_2^2}, \quad P_{12} = \frac{\sqrt{|\alpha_1\alpha_2|}}{1 - \lambda_1\lambda_2}. \tag{14}$$

It is obvious that $P_1 > 0$, $P_2 > 0$, and $P_{12} > 0$. Without loss of generality, we assume $P_1 \geq P_2 > 0$. We first determine the initial realization $(\boldsymbol{A}_0, \boldsymbol{b}_0, \boldsymbol{c}_0, d_0)$ as follows:

$$\left[\begin{array}{c|c} \boldsymbol{A}_0 & \boldsymbol{b}_0 \\ \hline \boldsymbol{c}_0 & d_0 \end{array}\right] = \left[\begin{array}{cc|c} \lambda_1 & 0 & \sqrt{|\alpha_1|} \\ 0 & \lambda_2 & \sqrt{|\alpha_2|} \\ \hline \sigma_1\sqrt{|\alpha_1|} & \sigma_2\sqrt{|\alpha_2|} & d \end{array}\right] \tag{15}$$

where $\sigma_1 = \mathrm{sign}(\alpha_1)$ and $\sigma_2 = \mathrm{sign}(\alpha_2)$. The controllability Gramian $\boldsymbol{K}_0$ and the observability Gramian $\boldsymbol{W}_0$ of the initial realization $(\boldsymbol{A}_0, \boldsymbol{b}_0, \boldsymbol{c}_0, d_0)$ are given by

$$\boldsymbol{K}_0 = \begin{bmatrix} P_1 & P_{12} \\ P_{12} & P_2 \end{bmatrix}, \quad \boldsymbol{W}_0 = \begin{bmatrix} P_1 & \sigma_1\sigma_2 P_{12} \\ \sigma_1\sigma_2 P_{12} & P_2 \end{bmatrix}. \tag{16}$$

The synthesis method of the balanced realization depends on the signatures σ_1 and σ_2. We consider two cases of the signatures: 1)$\sigma_1 = \sigma_2$ and 2)$\sigma_1 \neq \sigma_2$.

1) The case of $\sigma_1 = \sigma_2$

In this case, the controllability and observability Gramians are equal as follows:

$$\boldsymbol{K}_0 = \boldsymbol{W}_0 = \begin{bmatrix} P_1 & P_{12} \\ P_{12} & P_2 \end{bmatrix}. \tag{17}$$

We can diagonalize the Gramians $\boldsymbol{K}_0$ and $\boldsymbol{W}_0$ by the coordinate transformation by an orthogonal matrix $\boldsymbol{U}$ given by

$$\boldsymbol{U} = \begin{bmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{bmatrix}. \tag{18}$$

Under the coordinate transformation by the orthogonal matrix $\boldsymbol{U}$, the Gramians $\boldsymbol{K}_0$ and $\boldsymbol{W}_0$ are transformed into $\overline{\boldsymbol{K}}_0 = \boldsymbol{U}^T\boldsymbol{K}_0\boldsymbol{U}$ and $\overline{\boldsymbol{W}}_0 = \boldsymbol{U}^T\boldsymbol{K}_0\boldsymbol{U}$. The Gramians $\overline{\boldsymbol{K}}_0$ and $\overline{\boldsymbol{W}}_0$ are equal as is (19) at the bottom of the next page. The Gramians $\overline{\boldsymbol{K}}_0$ and $\overline{\boldsymbol{W}}_0$ are seen to be diagonal if we specify the parameter ϕ as

$$\phi = -\frac{1}{2}\tan^{-1}\left(\frac{2P_{12}}{P_1 - P_2}\right), \quad -\frac{\pi}{4} \leq \phi < 0. \tag{20}$$

Since we assume $P_1 \geq P_2$, the range of the argument of the function $\tan^{-1}$ is $(0, \infty]$, which yields the range of ϕ given in (20). Substituting (20) into (19) yields the controllability Gramian $\boldsymbol{K}_\mathrm{b}$ and the observability Gramian $\boldsymbol{W}_\mathrm{b}$ as follows:

$$\boldsymbol{K}_{\mathrm{b}} = \boldsymbol{W}_{\mathrm{b}} = \mathrm{diag}(\theta_1, \theta_2) \tag{21}$$
$$\theta_1 = \frac{1}{2}(P_1 + P_2) + \frac{1}{2}\sqrt{(P_1 - P_2)^2 + 4P_{12}^2} \tag{22}$$
$$\theta_2 = \frac{1}{2}(P_1 + P_2) - \frac{1}{2}\sqrt{(P_1 - P_2)^2 + 4P_{12}^2}. \tag{23}$$

The explicit expression of the balanced realization $(\boldsymbol{A}_{\mathrm{b}}, \boldsymbol{b}_{\mathrm{b}}, \boldsymbol{c}_{\mathrm{b}}, d_{\mathrm{b}})$ is given by (24) at the bottom of the page, where $\sigma = \sigma_1 = \sigma_2$. We can confirm that the coefficient matrices $(\boldsymbol{A}_{\mathrm{b}}, \boldsymbol{b}_{\mathrm{b}}, \boldsymbol{c}_{\mathrm{b}}, d_{\mathrm{b}})$ satisfy the following symmetry properties:

$$\boldsymbol{A}_{\mathrm{b}}^T = \boldsymbol{\Sigma}\boldsymbol{A}_{\mathrm{b}}\boldsymbol{\Sigma}, \quad \boldsymbol{c}_{\mathrm{b}}^T = \boldsymbol{\Sigma}\boldsymbol{b}_{\mathrm{b}}, \quad \boldsymbol{\Sigma} = \pm\boldsymbol{I}. \tag{25}$$

2) The case of $\sigma_1 = -\sigma_2$

In this case, the controllability and observability Gramians are given by

$$\boldsymbol{K}_0 = \begin{bmatrix} P_1 & P_{12} \\ P_{12} & P_2 \end{bmatrix}, \quad \boldsymbol{W}_0 = \begin{bmatrix} P_1 & -P_{12} \\ -P_{12} & P_2 \end{bmatrix}. \tag{26}$$

We can diagonalize the Gramians $\boldsymbol{K}_0$ and $\boldsymbol{W}_0$ by the coordinate transformation matrix $\boldsymbol{T}$ given by

$$\boldsymbol{T} = \begin{bmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{bmatrix}. \tag{27}$$

Under the coordinate transformation by the nonsingular matrix $\boldsymbol{T}$, the Gramians $\boldsymbol{K}_0$ and $\boldsymbol{W}_0$ are transformed into $\overline{\boldsymbol{K}}_0$ and $\overline{\boldsymbol{W}}_0$, as in (28) and (29) at the bottom of the page. The Gramians $\overline{\boldsymbol{K}}_0$ and $\overline{\boldsymbol{W}}_0$ are equal and diagonal if we specify the parameter t as

$$t = \frac{1}{2}\tanh^{-1}\left(\frac{2P_{12}}{P_1 + P_2}\right) > 0. \tag{30}$$

Substituting (30) into (28) and (29) yields the controllability Gramian $\boldsymbol{K}_{\mathrm{b}}$ and the observability Gramian $\boldsymbol{W}_{\mathrm{b}}$ as follows:

$$\boldsymbol{K}_{\mathrm{b}} = \boldsymbol{W}_{\mathrm{b}} = \mathrm{diag}(\theta_1, \theta_2) \tag{31}$$
$$\theta_1 = \frac{1}{2}\sqrt{(P_1 + P_2)^2 - 4P_{12}^2} + \frac{1}{2}(P_1 - P_2) \tag{32}$$
$$\theta_2 = \frac{1}{2}\sqrt{(P_1 + P_2)^2 - 4P_{12}^2} - \frac{1}{2}(P_1 - P_2). \tag{33}$$

The explicit expression of the balanced realization $(\boldsymbol{A}_{\mathrm{b}}, \boldsymbol{b}_{\mathrm{b}}, \boldsymbol{c}_{\mathrm{b}}, d_{\mathrm{b}})$ is given by (34) at the bottom of the next page. We can confirm that the coefficient matrices $(\boldsymbol{A}_{\mathrm{b}}, \boldsymbol{b}_{\mathrm{b}}, \boldsymbol{c}_{\mathrm{b}}, d_{\mathrm{b}})$ satisfy the following symmetry properties:

$$\boldsymbol{A}_{\mathrm{b}}^T = \boldsymbol{\Sigma}\boldsymbol{A}_{\mathrm{b}}\boldsymbol{\Sigma}, \quad \boldsymbol{c}_{\mathrm{b}}^T = \boldsymbol{\Sigma}\boldsymbol{b}_{\mathrm{b}}, \quad \boldsymbol{\Sigma} = \pm\mathrm{diag}(1, -1). \tag{35}$$

B. Two Poles Are Real and Multiple $\left(q_1^2 - 4q_2 = 0\right)$

We consider second-order digital filters whose poles are real and multiple as follows:

$$H(z) = \frac{\alpha_1}{z - \lambda} + \frac{\alpha_2}{(z - \lambda)^2} + d \tag{36}$$

where λ is a real double pole, and (α_1, α_2) are *real* scalars. We define the scalar parameters Q_1, Q_2, and Q_{12} as follows:

$$Q_1 = \frac{\alpha_1^2}{4|\alpha_2|}\frac{1}{1 - \lambda^2} + \sigma\alpha_1\frac{\lambda}{(1 - \lambda^2)^2} + |\alpha_2|\frac{1 + \lambda^2}{(1 - \lambda)^3} \tag{37}$$
$$Q_2 = |\alpha_2|\frac{1}{1 - \lambda^2} \tag{38}$$
$$Q_{12} = \frac{1}{2}\sigma\alpha_1\frac{1}{1 - \lambda^2} + |\alpha_2|\frac{\lambda}{(1 - \lambda^2)^2} \tag{39}$$

$$\overline{\boldsymbol{K}}_0 = \overline{\boldsymbol{W}}_0 = \begin{bmatrix} \frac{1}{2}(P_1 + P_2) + \frac{1}{2}(P_1 - P_2)\cos(2\phi) - P_{12}\sin(2\phi) & \frac{1}{2}(P_1 - P_2)\sin(2\phi) + P_{12}\cos(2\phi) \\ \frac{1}{2}(P_1 - P_2)\sin(2\phi) + P_{12}\cos(2\phi) & \frac{1}{2}(P_1 + P_2) - \frac{1}{2}(P_1 - P_2)\cos(2\phi) + P_{12}\sin(2\phi) \end{bmatrix} \tag{19}$$

$$\begin{aligned} \left[\begin{array}{c|c} \boldsymbol{A}_{\mathrm{b}} & \boldsymbol{b}_{\mathrm{b}} \\ \hline \boldsymbol{c}_{\mathrm{b}} & d_{\mathrm{b}} \end{array}\right] &= \left[\begin{array}{c|c} \boldsymbol{U}^T\boldsymbol{A}_0\boldsymbol{U} & \boldsymbol{U}^T\boldsymbol{b}_0 \\ \hline \boldsymbol{c}_0\boldsymbol{U} & d_0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} \frac{1}{2}(\lambda_1 + \lambda_2) + \frac{1}{2}(\lambda_1 - \lambda_2)\cos(2\phi) & \frac{1}{2}(\lambda_1 - \lambda_2)\sin(2\phi) & \sqrt{|\alpha_1|}\cos(\phi) - \sqrt{|\alpha_2|}\sin(\phi) \\ \frac{1}{2}(\lambda_1 - \lambda_2)\sin(2\phi) & \frac{1}{2}(\lambda_1 + \lambda_2) - \frac{1}{2}(\lambda_1 - \lambda_2)\cos(2\phi) & \sqrt{|\alpha_1|}\sin(\phi) - \sqrt{|\alpha_2|}\cos(\phi) \\ \hline \sigma\left(\sqrt{|\alpha_1|}\cos(\phi) - \sqrt{|\alpha_2|}\sin(\phi)\right) & \sigma\left(\sqrt{|\alpha_1|}\sin(\phi) - \sqrt{|\alpha_2|}\cos(\phi)\right) & d \end{array}\right] \end{aligned} \tag{24}$$

$$\begin{aligned} \overline{\boldsymbol{K}}_0 &= \boldsymbol{T}^{-1}\boldsymbol{K}_0\boldsymbol{T}^{-T} \\ &= \begin{bmatrix} \frac{1}{2}(P_1 + P_2)\cosh(2t) - P_{12}\sinh(2t) + \frac{1}{2}(P_1 - P_2) & -\frac{1}{2}(P_1 + P_2)\sinh(2t) + P_{12}\cosh(2t) \\ -\frac{1}{2}(P_1 + P_2)\sinh(2t) + P_{12}\cosh(2t) & \frac{1}{2}(P_1 + P_2)\cosh(2t) - P_{12}\sinh(2t) - \frac{1}{2}(P_1 - P_2) \end{bmatrix} \end{aligned} \tag{28}$$
$$\begin{aligned} \overline{\boldsymbol{W}}_0 &= \boldsymbol{T}^T\boldsymbol{W}_0\boldsymbol{T} \\ &= \begin{bmatrix} \frac{1}{2}(P_1 + P_2)\cosh(2t) - P_{12}\sinh(2t) + \frac{1}{2}(P_1 - P_2) & \frac{1}{2}(P_1 + P_2)\sinh(2t) - P_{12}\cosh(2t) \\ \frac{1}{2}(P_1 + P_2)\sinh(2t) - P_{12}\cosh(2t) & \frac{1}{2}(P_1 + P_2)\cosh(2t) - P_{12}\sinh(2t) - \frac{1}{2}(P_1 - P_2) \end{bmatrix} \end{aligned} \tag{29}$$

where $\sigma = \mathrm{sign}(\alpha_2)$. We first determine the initial realization $(\boldsymbol{A}_0, \boldsymbol{b}_0, \boldsymbol{c}_0, d_0)$ as follows:

$$\left[\begin{array}{c|c} \boldsymbol{A}_0 & \boldsymbol{b}_0 \\ \hline \boldsymbol{c}_0 & d_0 \end{array}\right] = \left[\begin{array}{cc|c} \lambda & 1 & \sigma\frac{\alpha_1}{2\sqrt{|\alpha_2|}} \\ 0 & \lambda & \sqrt{|\alpha_2|} \\ \hline \sigma\sqrt{|\alpha_2|} & \frac{\alpha_1}{2\sqrt{|\alpha_2|}} & d \end{array}\right]. \quad (40)$$

The controllability Gramian $\boldsymbol{K}_0$ and the observability Gramian $\boldsymbol{W}_0$ of the initial realization $(\boldsymbol{A}_0, \boldsymbol{b}_0, \boldsymbol{c}_0, d_0)$ are given by

$$\boldsymbol{K}_0 = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12} & Q_2 \end{bmatrix}, \quad \boldsymbol{W}_0 = \begin{bmatrix} Q_2 & Q_{12} \\ Q_{12} & Q_1 \end{bmatrix}. \quad (41)$$

We can diagonalize the Gramians $\boldsymbol{K}_0$ and $\boldsymbol{W}_0$ by the coordinate transformation matrix $\boldsymbol{T}$ given by

$$\boldsymbol{T} = \frac{1}{\sqrt{2}} \begin{bmatrix} \tau & -\tau \\ \tau^{-1} & \tau^{-1} \end{bmatrix}. \quad (42)$$

Under the coordinate transformation by the nonsingular matrix $\boldsymbol{T}$, the Gramians $\boldsymbol{K}_0$ and $\boldsymbol{W}_0$ are transformed into $\overline{\boldsymbol{K}}_0$ and $\overline{\boldsymbol{W}}_0$ such as (43) and (44) at the bottom of the page. The Gramians $\overline{\boldsymbol{K}}_0$ and $\overline{\boldsymbol{W}}_0$ are equal and diagonal if we specify the parameter τ as

$$\tau = \left(\frac{Q_1}{Q_2}\right)^{1/4}. \quad (45)$$

Substituting (45) into (43) and (44) yields the controllability Gramian $\boldsymbol{K}_\mathrm{b}$ and the observability Gramian $\boldsymbol{W}_\mathrm{b}$ as follows:

$$\boldsymbol{K}_\mathrm{b} = \boldsymbol{W}_\mathrm{b} = \mathrm{diag}(\theta_1, \theta_2) \quad (46)$$
$$\theta_1 = \sqrt{Q_1 Q_2} + Q_{12} \quad (47)$$
$$\theta_2 = \sqrt{Q_1 Q_2} - Q_{12}. \quad (48)$$

The explicit expression of the balanced realization $(\boldsymbol{A}_\mathrm{b}, \boldsymbol{b}_\mathrm{b}, \boldsymbol{c}_\mathrm{b}, d_\mathrm{b})$ is given by (49) at the bottom of the page. We can confirm that the coefficient matrices $(\boldsymbol{A}_\mathrm{b}, \boldsymbol{b}_\mathrm{b}, \boldsymbol{c}_\mathrm{b}, d_\mathrm{b})$ satisfy the following symmetry properties:

$$\boldsymbol{A}_\mathrm{b}^T = \boldsymbol{\Sigma}\boldsymbol{A}_\mathrm{b}\boldsymbol{\Sigma}, \quad \boldsymbol{c}_\mathrm{b}^T = \boldsymbol{\Sigma}\boldsymbol{b}_\mathrm{b}, \quad \boldsymbol{\Sigma} = \pm\mathrm{diag}(1, -1). \quad (50)$$

IV. Conclusions

This letter has discussed the synthesis of balanced realizations of second-order digital filters with *real poles*. We derived explicit expressions of the balanced realizations of these types of second-order digital filters. We can compute the coefficient matrices of the balanced realizations directly from the transfer function. The results are quite useful, especially when we synthesize the minimum L_2-sensitivity realization since it requires the closed-form expression of the balanced realization.

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$$\left[\begin{array}{c|c} \boldsymbol{A}_\mathrm{b} & \boldsymbol{b}_\mathrm{b} \\ \hline \boldsymbol{c}_\mathrm{b} & d_\mathrm{b} \end{array}\right] = \left[\begin{array}{c|c} \boldsymbol{T}^{-1}\boldsymbol{A}_0\boldsymbol{T} & \boldsymbol{T}^{-1}\boldsymbol{b}_0 \\ \hline \boldsymbol{c}_0\boldsymbol{T} & d_0 \end{array}\right]$$
$$= \left[\begin{array}{cc|c} \frac{1}{2}(\lambda_1+\lambda_2) + \frac{1}{2}(\lambda_1-\lambda_2)\cosh(2t) & \frac{1}{2}(\lambda_1-\lambda_2)\sinh(2t) & \sqrt{|\alpha_1|}\cosh(t) - \sqrt{|\alpha_2|}\sinh(t) \\ -\frac{1}{2}(\lambda_1-\lambda_2)\sinh(2t) & \frac{1}{2}(\lambda_1+\lambda_2) - \frac{1}{2}(\lambda_1-\lambda_2)\cosh(2t) & -\sqrt{|\alpha_1|}\sinh(t) + \sqrt{|\alpha_2|}\cosh(t) \\ \hline \sigma_1\sqrt{|\alpha_1|}\cosh(t) + \sigma_2\sqrt{|\alpha_2|}\sinh(t) & \sigma_1\sqrt{|\alpha_1|}\sinh(t) + \sigma_2\sqrt{|\alpha_2|}\cosh(t) & d \end{array}\right] \quad (34)$$

$$\overline{\boldsymbol{K}}_0 = \boldsymbol{T}^{-1}\boldsymbol{K}_0\boldsymbol{T}^{-T}$$
$$= \frac{1}{2}\begin{bmatrix} \tau^{-2}Q_1 + \tau^2 Q_2 + 2Q_{12} & -\tau^{-2}Q_1 + \tau^2 Q_2 \\ -\tau^{-2}Q_1 + \tau^2 Q_2 & \tau^{-2}Q_1 + \tau^2 Q_2 - 2Q_{12} \end{bmatrix} \quad (43)$$
$$\overline{\boldsymbol{W}}_0 = \boldsymbol{T}^T\boldsymbol{W}_0\boldsymbol{T}$$
$$= \frac{1}{2}\begin{bmatrix} \tau^{-2}Q_1 + \tau^2 Q_2 + 2Q_{12} & \tau^{-2}Q_1 - \tau^2 Q_2 \\ \tau^{-2}Q_1 - \tau^2 Q_2 & \tau^{-2}Q_1 + \tau^2 Q_2 - 2Q_{12} \end{bmatrix} \quad (44)$$

$$\left[\begin{array}{c|c} \boldsymbol{A}_\mathrm{b} & \boldsymbol{b}_\mathrm{b} \\ \hline \boldsymbol{c}_\mathrm{b} & d_\mathrm{b} \end{array}\right] = \left[\begin{array}{cc|c} \lambda + \frac{1}{2}\tau^{-2} & \frac{1}{2}\tau^{-2} & \sigma\frac{\alpha_1}{2\sqrt{2|\alpha_2|}}\tau^{-1} + \sqrt{\frac{|\alpha_2|}{2}}\tau \\ -\frac{1}{2}\tau^{-2} & \lambda - \frac{1}{2}\tau^{-2} & -\sigma\frac{\alpha_1}{2\sqrt{2|\alpha_2|}}\tau^{-1} + \sqrt{\frac{|\alpha_2|}{2}}\tau \\ \hline \frac{\alpha_1}{2\sqrt{2|\alpha_2|}}\tau^{-1} + \sigma\sqrt{\frac{|\alpha_2|}{2}}\tau & \frac{\alpha_1}{2\sqrt{2|\alpha_2|}}\tau^{-1} - \sigma\sqrt{\frac{|\alpha_2|}{2}}\tau & d \end{array}\right] \quad (49)$$